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Commutators on Characteristic Surfaces*

ARTHUR KOMAR
Belfer Graduate School of Science, Yeshiva University, New York, New York
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The Lagrangian formalism is employed to derive the commutation relations on null surfaces for relativistic field theories. The theories treated are the Klein-Gordon field, the Maxwell field, the linearized gravitation theory, and the general theory of relativity. Special attention is paid to the treatment of null surfaces at infinity, on which we are able to obtain the commutation relations for the "news function," which represents the independent radiation modes of the field. For the general theory of relativity, the methods of this paper seem appropriate only when we truncate the theory by excluding solutions which are not asymptotically flat in the sense of Penrose.

I. INTRODUCTION

THE program for the quantization of the gravitational field has long been plagued by the inability to construct a complete, nonredundant set of true observables within the classical Einstein theory, due essentially to the presence of four (nonlinear) constraints among the ten Einstein field equations. In recent years, Penrose has indicated[1] how, by focusing our attention on null surfaces rather than the traditional space-like surfaces, the difficulty of the constraints could be ignored, and solutions of the field equations could be characterized to a large extent by a single complex function constructed by projecting the Riemann tensor into the null surface. In a recent paper,[2] Penrose showed that for asymptotically flat surfaces a particularly appropriate choice of null surface would be the null cone at infinity. For such a choice of null surface the complex scalar function which essentially characterizes the Riemann-Einstein manifold is closely related to the Bondi "news function."[3]

It would thus appear that Penrose's scalar is ideally suited for the description of gravitational radiation and therefore particularly appropriated for use as a basis for the construction of a quantum theory. An additional advantage obtained by working in the neighborhood of infinity is that the nonlinear terms in the Einstein field equations may be regarded as vanishingly small and we may expect that relations derived for the linearized theory of gravitation would continue to be valid in the full theory. A major obstacle to the construction of Poisson brackets for the Penrose scalar is the essential use of null surfaces in its definition. The usual canonical formalism presupposes canonical variables defined on a space-like hypersurface. It is decidedly inappropriate for generalization to a null hypersurface, for the naturally defined canonical momentum generally turns out to be a constraint within the null surface. For familiar theories one could in principle propagate the conventional commutation relations from the initial space-like surface to the desired null surface and thereby discover an equivalent set of commutation relations for variables defined on a null surface. Purely apart from considerations of feasibility, we are not interested in such an approach since it is precisely for the situation where we have difficulty in constructing commutators of observables on space-like hypersurfaces that we are motivated to investigate commutators defined on null surfaces. We shall therefore employ the less familiar but more covariant Lagrangian formalism[4] in all our considerations.

In order to introduce only one difficulty at a time, we shall divide the presentation into several stages. Section II will present a brief review of the Lagrangian formalism and as a simple illustration we shall treat the free particle. In Sec. III we shall illustrate how to apply the Lagrangian formalism to obtain the usual equal time commutation relations for the Klein-Gordon field. In Sec. IV we shall observe the nature of the additional complications introduced when we employ characteristic surfaces. We shall derive the commutators of the Klein-Gordon field on a null cone and at null infinity. Section V will be devoted to the treatment of the Maxwell field, both on a space-like and on a null surface hypersurface, and at null infinity. The new difficulties encountered at this stage are the presence of a gauge group, as well as the need to introduce explicitly the Penrose scalar. Section VI will, in a similar fashion, treat the linearized theory of gravitation. In the concluding section, VII, we shall discuss the relevance of the results of Sec. VI for the general theory of relativity.

II. THE LAGRANGIAN FORMALISM

Within the Hamiltonian formalism, a canonical transformation may be defined as a transformation of the canonical variables (position and momentum) in phase space which preserves Hamilton's equations of motion. The canonical transformations may be shown to form a group. Any arbitrary function of the canonical variables can be shown to generate an infinitesimal canonical transformation. The Poisson bracket of any

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[2] R. Penrose, Phys. Rev. Letters **10**, 66 (1963).

[3] H. Bondi, M. vander Burg, and A. Metzner, Proc. Roy. Soc. (London) **A269**, 21 (1962).

[4] P. G. Bergmann and R. Schiller, Phys. Rev. **89**, 4 (1953).

two functions of the canonical variables, $f(q_i,p_j)$ and $g(q_i,p_j)$, defined as

$$[f,g]=\sum_i\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) \tag{1}$$

is found to equal the change in f induced by the infinitesimal canonical transformation generated by g (or equivalently the negative of the change induced in g by the infinitesimal canonical transformation generated by f). A further, deeper understanding of the Poisson bracket is obtained by employing the Jacobi identity

$$[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0, \tag{2}$$

which is a direct consequence of the definition Eq. (1). From Eq. (2) we can easily conclude that $[f,g]$, which is again some function of the canonical variables, generates the infinitesimal canonical transformation which is the commutator of the infinitesimal canonical transformations generated by f and g individually. The canonical group contains a normal subgroup, the invariant canonical group obtained by considering those canonical transformations which leave invariant the form of the Hamiltonian as a function of the canonical variables. It is evident that the infinitesimal invariant canonical transformations are generated by those functions of the canonical variables which are constants of the motion.

Within the Lagrangian formalism, a canonical transformation may be defined as a transformation of the dynamical variables of configuration space with the property that, apart from the addition of a total time derivative to the Lagrangian, it leaves invariant the maximal order of time derivatives occurring in the Lagrangian. For the usual theories under consideration in this paper the Lagrangian is a function of only the positions and velocities. The canonicity condition then asserts that the transformation does not induce acceleration dependent terms into the Lagrangian. In view of the fact that the definition of a canonical transformation now requires explicit reference to the form of the Lagrangian, and that under canonical transformations the form of the Lagrangian in general will change, it is rather evident that the canonical transformations do not form a group in configuration space. However, if we consider only those canonical transformations which leave invariant the form of the Lagrangian as a function of the positions and velocities, they do form a group, and in fact coincide with the group of invariant canonical transformations as defined in the Hamiltonian formalism.

The precise relationship between the constant of the motion, C, which generates an infinitesimal invariant canonical transformation and the infinitesimal change in the dynamical variable, δq_a, which it induces is found to be[4]

$$\frac{dC}{dt}+\sum_a \delta q_a F^a=0, \tag{3}$$

where F^a are the Euler-Lagrange equations derived from the Lagrangian under consideration. We can therefore read directly from the coefficient of F^a the Poisson bracket of q_a and C:

$$\delta q_a=[q_a,C]. \tag{4}$$

As an elementary illustration of this method for determining Poisson brackets, consider the free particle given by the Lagrangian

$$L=\tfrac{1}{2}\sum_a m\dot{X}_a^2. \tag{5}$$

The Euler-Lagrange equations for this Lagrangian are

$$F_a\equiv -md^2X_a/dt^2=0. \tag{6}$$

Thus if we define

$$C=\sum_a \lambda_a(X_a-\dot{X}_a t), \tag{7}$$

where the λ_a are a set of arbitrary constants, we readily observe that as a consequence of the equations of motion (6), C is a constant of the motion. In fact

$$0=\frac{dC}{dt}+\sum_a \lambda_a d^2X_a/dt^2 t=\frac{dC}{dt}+\sum_a\left(\frac{-\lambda_a t}{m}\right)F_a. \tag{8}$$

Comparing with Eqs. (3) and (4) we find

$$\frac{-\lambda_a t}{m}=[X_a,C]=[X_a,\sum_b \lambda_b(X_b-\dot{X}_b t)]. \tag{9}$$

Since Eq. (9) must be valid for all times t and for all values of the arbitrary constants λ_a, we readily conclude that

$$[X_a,X_b]=0,\quad [X_a,m\dot{X}_b]=\delta_{ab}, \tag{10}$$

in full agreement with the initial definition, Eq. (1).

III. THE KLEIN-GORDON FIELD

We now wish to apply the Lagrangian formalism to the problem of determining the Poisson brackets for a classical field. It is still true that the invariant canonical transformations are generated by constants of the motion. Thus, Eq. (4) remains valid, where now the constant of the motion C may be obtained by integrating a conserved vector density C^ρ over a space-like hypersurface:

$$C\equiv\int C^\rho dS_\rho. \tag{11}$$

(A summation convention will be understood on repeated space-time indices.) The relationship between the constant of the motion C and the change in the field variable δq_a which it generates is obtained by the evident modification of Eq. (3)[4]

$$C^\rho{}_{,\rho}+\sum_a \delta q_a F^a=0, \tag{12}$$

where, as before, F^a are the Euler-Lagrange (field) equations for the Lagrangian under consideration. (A comma denotes differentiation. Space-time indices may be raised and lowered by means of the Minkowski metric, with signature 1, −1, −1, −1.)

Let us now consider the Lagrangian for the scalar field, Φ:

$$L=\tfrac{1}{2}\Phi_{,\rho}\Phi^{,\rho}. \tag{13}$$

(We could, if we wish, include a mass term. But this would not affect our discussion in any essential way.) The field equation for this Lagrangian is

$$F\equiv-\Phi_{,\rho}{}^{\rho}=0. \tag{14}$$

We shall define α to be an arbitrary solution of the equation

$$\alpha_{,\rho}{}^{\rho}=0. \tag{15}$$

We distinguish between Eqs. (14) and (15), although they are really the same equation, in order to emphasize that Φ is to be regarded as a dynamical variable satisfying commutation relations, whereas α is simply an arbitrary but specific mathematical function which happens to be a solution of Eq. (15). Since Eq. (15) is second order in time, it is evident that both α and $\partial\alpha/\partial t$ may be given as independent arbitrary functions on an initial space-like hypersurface.

We employ α in order to define the vector field C^ρ:

$$C^\rho=\alpha^\rho\Phi-\alpha\Phi^\rho. \tag{16}$$

Taking the divergence of Eq. (16) we find by virtue of Eqs. (14) and (15)

$$C^\rho{}_{,\rho}+(-\alpha)F=0. \tag{17}$$

Comparing Eq. (17) with Eq. (12), we see from Eqs. (4) and (11) that

$$\left[\Phi,\int C^\rho dS_\rho\right]=-\alpha. \tag{18}$$

Note that Eq. (15) assures that $\delta\Phi$ is a solution of the field equation (14).

On the initial space-like hypersurface $t=$const; Eq. (18) yields (the index S running from 1 to 3):

$$\left[\Phi(x^s,0),\int\left(\frac{\partial\alpha}{\partial t}(y^s,0)\Phi(y^s,0)-\alpha(y^s,0)\frac{\partial\Phi}{\partial t}(y^s,0)\right)d_3y\right]=-\alpha(x^s,0). \tag{19}$$

In view of the fact that $\alpha(y^s,0)$ and $(\partial\alpha/\partial t)(y^s,0)$ are independent arbitrary functions, Eq. (19) can only be valid if

$$\begin{gathered}[\Phi(x^s,0),\Phi(y^s,0)]=0,\\ [\Phi(x^s,0),(\partial\Phi/\partial t)(y^s,0)]=\delta_3(x^s-y^s).\end{gathered} \tag{20}$$

This is most readily obtained by assigning in Eq. (19) alternatively $\alpha(y^s,0)=0$, $(\partial\alpha/\partial t)(y^s,0)=\delta_3(y^s-z^s)$, and $\alpha(y^s,0)=\delta_3(y^s-z^s)$, $(\partial\alpha/\partial t)(y^s,0)=0$. We see, incidently, from these assignments that we encounter no difficulties concerning the existence or convergence of the integral defined in Eq. (11).

IV. COMMUTATORS ON NULL CONES

In order to treat the scalar field on a null cone much of the development of the previous section may be taken over intact. Selecting the Lagrangian as in Eq. (13), and defining the auxiliary weight function, α, as a solution of Eq. (15), we can construct the conserved vector field of Eq. (16). The commutation relation, Eq. (18), follows eactly as in the case of the space-like hypersurface. However, there are two important changes in the subsequent development which result from the use of null cones. (1) In the integral of Eq. (11) the surface is chosen to be the null surface. We must be particularly careful about handling the end points of the integral if C is to be a constant of the motion. In point of fact the additional caution will only be required in our treatment of the scalar field and will be presented in detail. For the Maxwell field and the gravitational field we shall find it possible to select an integrand in Eq. (11) just as localized on the null cone as it was on the space-like hypersurface in Sec. III. (2) Since the null cone is a characteristic surface for the d'Alembert equation, Eq. (15), it is no longer true that both α and its first derivative off the initial surface can be prescribed independently. It is still true, however, that α itself can be arbitrarily prescribed on the null cone, and that shall turn out to be sufficient for our purpose.

It will be convenient to introduce coordinates adapted to the null cone which is to be the initial surface. We therefore take as our coordinates

$$\begin{aligned}X^0&=u=(1/\sqrt{2})(t-r), & X^2&=\theta\\ X^1&=v=(1/\sqrt{2})(t+r), & X^3&=\phi\end{aligned} \tag{21}$$

where r, θ, ϕ are the usual polar coordinates. In this coordinate system the Minkowski metric becomes

$$g_{\mu\nu}=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&-r^2&0\\0&0&0&-r^2\sin^2\theta\end{bmatrix}. \tag{22}$$

When employing these coordinates we understand the comma to denote covariant differentiation with respect to this metric.

The initial surface with which we shall be concerned satisfies the equation

$$X^0\equiv u=k, \tag{23}$$

where k is some fixed, but arbitrary constant. The surface element of this null cone is evidently

$$dS_\rho=\delta_\rho{}^0r^2\sin\theta dvd\theta d\phi. \tag{24}$$

Thus from Eqs. (11) and (16), we find

$$C=\int\left(\frac{\partial\alpha}{\partial v}\Phi-\alpha\frac{\partial\Phi}{\partial v}\right)r^2\sin\theta dv d\theta d\phi$$

$$=\int\left(2\frac{\partial(r\alpha)}{\partial v}r\Phi-\frac{\partial(r^2\Phi\alpha)}{\partial v}\right)\sin\theta dv d\theta d\phi. \quad (25)$$

We note at this point two interesting properties of Eq. (25): (1) In contrast with the case of the space-like hypersurface, projection of the integrand into the null surface element yields derivatives exclusively *within* the null surface, rather than off of the surface. (2) The second term in the integrand, being a perfect differential, will yield contributions exclusively from the end points. We shall denote these end points symbolically by 0 and ∞. We can, if we wish, take the arbitrary function α to vanish at the end points and thereby avoid this complication. Although such a choice will in fact be made for the Maxwell and the gravitational fields, it will prove convenient in the case of the scalar field to assume only that the behavior of $\Phi\alpha$ is such that the contributions from the end points are nonsingular. What will essentially be entailed is that Φ fall off no slower than r^{-1} at infinity, and diverge no worse than r^{-1} at the origin—not particularly severe restrictions.

In view of the arbitrariness of α on the initial surface we may take it to be

$$\alpha=\sigma(v-v')\delta(\Omega-\Omega'), \quad (26)$$

where σ is the unit antisymmetric step function, i.e.,

$$\sigma(x)=\begin{matrix}-\frac{1}{2} & X<0,\\ +\frac{1}{2} & X>0,\end{matrix} \quad (27)$$

and $\delta(\Omega-\Omega')$ is the Dirac delta function on the unit sphere, Ω denoting the solid angle, i.e.,

$$\delta(\Omega-\Omega')=(\sin\theta)^{-1}\delta(\theta-\theta')\delta(\phi-\phi'). \quad (28)$$

Inserting Eq. (26) into Eq. (25) we obtain

$$C=2r'\Phi(v',\Omega')-\tfrac{1}{2}[r\Phi](\infty,\Omega')-\tfrac{1}{2}[r\Phi](0,\Omega') \quad (29)$$

[where we have introduced the notation $\lim_{r\to a} r\Phi(r,\Omega)\equiv[r\Phi](a,\Omega)$]. Thus Eq. (18) yields

$$[\Phi(v,\Omega),\, 2r'\Phi(v',\Omega')-\tfrac{1}{2}([r\Phi](\infty,\Omega')+[r\Phi](0,\Omega'))]$$
$$=-[\sigma(v-v')\delta(\Omega-\Omega')/r]. \quad (30)$$

We cannot assume that the contributions from the end points commute with the field in the interior, since they can lie on the same null ray. However we may evaluate the commutator at the two end points. Thus at $r'=\infty$ we have

$$[\Phi(v,\Omega),\, \tfrac{3}{2}[r\Phi](\infty,\Omega')-\tfrac{1}{2}[r\Phi](0,\Omega')]$$
$$=-\delta(\Omega-\Omega')/2r, \quad (31)$$

while at $r'=0$ we have

$$[\Phi(v,\Omega),\, \tfrac{3}{2}(r\Phi)(0,\Omega')-\tfrac{1}{2}[r\Phi](\infty,\Omega')]$$
$$=+\delta(\Omega-\Omega')/2r. \quad (32)$$

If we now add Eqs. (31) and (32) we find

$$[\Phi(v,\Omega),\, [r\Phi](0,\Omega')+[r\Phi](\infty,\Omega')]=0. \quad (33)$$

Thus the sum of the contributions from the two end points does in fact commute with the field in the interior. Returning to Eq. (30) we can now write it in the form

$$[\Phi(v,\Omega),\Phi(v',\Omega)]=-\sigma(v-v')\delta(\Omega-\Omega')/2rr' \quad (34)$$

or, if we prefer

$$[\Phi(v,\Omega),\Phi(v',\Omega')]=-\sigma(r-r')\delta(\Omega-\Omega')/2rr' \quad (35)$$

since

$$\sigma(v-v')=\sigma(\sqrt{2}(r-r'))=\sigma(r-r'). \quad (36)$$

One can readily check that Eq. (35) is equivalent to the usual commutation relations, Eq. (20).

In order to obtain the commutation relations "at infinity," we must understand correctly what the appropriate limiting procedure is. If we wish to use the surface at past null infinity of Penrose,[2] the points of which are labeled by the coordinates v, θ, ϕ, we return to the equation of the initial surface Eq. (23) and go to the limit $k\to-\infty$, keeping v, θ, ϕ fixed. From Eqs. (21) it is evident that this limiting process implies $r\to\infty$ and $t\to-\infty$, thereby justifying the name "past null infinity." If we define

$$\rho(v,\Omega)\equiv\lim_{k\to-\infty} r\Phi(v,\Omega) \quad (37)$$

and employ this limit in Eq. (34), we obtain

$$[\rho(v,\Omega),\rho(v',\Omega')]=-\tfrac{1}{2}\sigma(v-v')\delta(\Omega-\Omega'). \quad (38)$$

Equation (38) is in agreement with the results of R. Sachs,[5] who obtained essentially the same expression by considering the commutation relations which one must impose on the data at infinity in order to obtain the usual commutation relations at finite points. The virtue of the present approach is that it can be extended to those cases where we do not know the correct commutation relations at finite points.

The limiting procedure of Eq. (37) clearly singles out the incoming radiation modes of the field. In order to obtain the commutators for the out-going radiation modes we could redo the entire analysis on a surface $v=k$, and then take the limit $k\to+\infty$. What we would obtain in this fashion is a relation analogous to Eq. (38) with v replaced everywhere by u, and $\rho(u,\Omega)$ defined by the appropriate modification of Eq. (37). It is of some interest to note that we could have obtained precisely the same commutation relations had we considered the

[5] R. Sachs, Phys. Rev. **128**, 2851 (1962).

massive Klein-Gordon field. The more interesting question of the commutators between the in-coming and the out-going radiation fields would of course require a complete solution of the field equations from $-\infty$ to $+\infty$, in which the value of the mass would need to enter. As long as we confine our attention to an initial surface, even though it may be null, the mass never need enter into our considerations.

V. THE MAXWELL FIELD

The Lagrangian for the electromagnetic field is

$$L=-\tfrac{1}{4}F_{\mu\gamma}F^{\mu\gamma}, \tag{39}$$

where

$$F_{\mu\gamma}=A_{\mu,\gamma}-A_{\nu,\mu}, \tag{40}$$

from which we readily deduce the field equations

$$F^{\mu}\equiv F^{\mu\lambda}{}_{,\lambda}=0. \tag{41}$$

Let us define the vector field C^{ρ} as

$$C^{\rho}\equiv\alpha_{\mu}F^{\mu\rho}+\beta_{\mu}F^{\mu\rho *}+\gamma_{,\mu}F^{\mu\rho}, \tag{42}$$

where $F^{\mu\rho *}$ is the dual of $F^{\mu\rho}$, that is,

$$F_{\mu\nu}{}^{*}\equiv\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}, \tag{43}$$

and $\epsilon_{\mu\nu\alpha\beta}$ is the totally antisymmetric tensor formed by the product $\sqrt{g}$ with the Levi-Civita tensor density. The distributions α_{μ}, β_{μ}, and γ are yet to be determined.

Taking the divergence of Eq. (42) we find from Eqs. (40), (41), and (43)

$$C^{\rho}{}_{,\rho}+(-\alpha_{\mu}-\gamma_{,\mu})F^{\mu}=(\alpha_{\mu,\nu}+\tfrac{1}{2}\beta^{\alpha,\beta}\epsilon_{\alpha\beta\mu\nu})F^{\mu\nu}. \tag{44}$$

If, analogous to Eq. (15), we require

$$\alpha_{\mu,\nu}-\alpha_{\nu,\mu}+\beta^{\alpha,\beta}\epsilon_{\alpha\beta\mu\nu}=0, \tag{45}$$

we satisfy Eq. (12) with

$$\delta A_{\mu}=-\alpha_{\mu}-\gamma_{,\mu}. \tag{46}$$

We should note that Eq. (45) assures that δA_{μ} will satisfy the field equations (41).

The potentials, A_{μ}, are of course defined only up to a gauge transformation

$$A'_{\mu}=A_{\mu}+a_{,\mu}. \tag{47}$$

If we wish, we can employ the arbitrary scalar field, a, to impose a subsidiary condition on the potentials. Alternatively, we can eliminate the arbitrary scalar field by working exclusively with the gauge-invariant field tensor of Eq. (40). From Eq. (46) we see that the completely unrestricted function γ reflects our ability to perform an arbitrary gauge transformation on the perturbed potentials δA_{μ}. We may select γ in order to preserve for the perturbed potentials the same subsidiary conditions which we imposed upon the original potentials, or alternatively, we can eliminate all reference to the arbitrary function γ by confining our attention to the perturbation of the field tensor. In this section we shall employ the latter course.

We have from Eqs. (4), (42), and (46)

$$\left[A_{\nu},\int(\alpha_{\mu}F^{\mu\rho}+\beta_{\mu}F^{\mu\rho *})dS_{\rho}\right]=-\alpha_{\nu}-\gamma_{,\nu}. \tag{48}$$

The last term in Eq. (42) was eliminated from the above integral by integrating by parts and employing the field equation (41). If we now take the curl of Eq. (48) we obtain the gauge-invariant commutation relations

$$\left[F_{\mu\nu},\int(\alpha_{\sigma}F^{\sigma\rho}+\beta_{\sigma}F^{\sigma\rho *})dS_{\rho}\right]=-\alpha_{\mu,\nu}+\alpha_{\nu,\mu}. \tag{49}$$

From Eqs. (43) and (45) we can also write

$$\left[F^{*}{}_{\mu\nu},\int(\alpha_{\sigma}F^{\sigma\rho}+\beta_{\sigma}F^{\sigma\rho *})dS_{\rho}\right]=-\beta_{\mu,\nu}+\beta_{\nu,\mu}. \tag{50}$$

If we confine our attention to a space-like hypersurface, which we take to be $t=$constant, the commutation relations for the field strengths are particularly transparent. This is due to the fact that α_{μ} and β_{μ} may both be chosen arbitrarily on a space-like hypersurface without contradiction of Eq. (45). By selecting alternately $\alpha_{\mu}=\delta_{\mu}{}^{\mu'}\delta(\mathbf{x}-\mathbf{x}')$, $\beta_{\mu}=0$ and $\alpha_{\mu}=0$, $\beta_{\mu}=\delta_{\mu}{}^{\mu'}\times\delta(\mathbf{x}-\mathbf{x}')$ we readily obtain

$$\begin{aligned}
&[F_{st}(\mathbf{x}),F^{r4}(\mathbf{x}')]=-\delta_{s}{}^{r}\delta_{,t}(\mathbf{x}-\mathbf{x}')+\delta_{t}{}^{r}\delta_{,s}(\mathbf{x}-\mathbf{x}'),\\
&[F_{st}(\mathbf{x}),F^{r4*}(\mathbf{x}')]=0,\\
&[F^{*}{}_{st}(\mathbf{x}),F^{r4}(\mathbf{x}')]=0,\\
&[F^{*}{}_{st}(\mathbf{x}),F^{r4*}(\mathbf{x}')]=-\delta_{s}{}^{4}\delta_{,t}(\mathbf{x}-\mathbf{x}')+\delta_{t}{}^{r}\delta_{,s}(\mathbf{x}-\mathbf{x}'),
\end{aligned} \tag{51}$$

where the indices r, s, t run from 1 to 3.

In order to obtain the commutation relations on a null surface, it will prove convenient to introduce a quadruple of null vectors adapted to the surface in question. Recall that in the coordinate system given by Eqs. (21) the null cone which we shall consider as our initial surface is given by Eq. (23). We shall define k_{μ} to be the gradient of the family of null surfaces which satisfy Eq. (23). If we define l_{μ} as the gradient of the family of surfaces $X'=v=$constant, we evidently have

$$k^{\mu}k_{\mu}=l^{\mu}l_{\mu}=0,\quad k^{\mu}l_{\mu}=1. \tag{52}$$

We complete the quadruple by introducing the complex conjugate pair of space-like null vectors m_{μ}, $\bar{m}_{\mu}$ which satisfy the conditions

$$k^{\mu}m_{\mu}=l^{\mu}m_{\mu}=m^{\mu}m_{\mu}=0,\quad m^{\mu}\bar{m}_{\mu}=-1. \tag{53}$$

In the specific coordinate system of Eqs. (21), these vectors can be chosen to have the form

$$k_{\mu}=\delta_{\mu}{}^{0},\quad l_{\mu}=\delta_{\mu}{}^{1},\quad m_{\mu}=(r/\sqrt{2})(\delta_{\mu}{}^{2}+i\sin\theta\delta_{\mu}{}^{3}) \tag{54}$$

or equivalently

$$k^{\mu}=\delta_{1}{}^{\mu},\quad l^{\mu}=\delta_{0}{}^{\mu},\quad m^{\mu}=\frac{-1}{r\sqrt{2}}\left(\delta_{2}{}^{\mu}+\frac{i}{\sin\theta}\delta_{3}{}^{\mu}\right). \tag{55}$$

A particularly useful set of identities which one can easily deduce from these expressions is

$$k^\mu l^\nu \epsilon_{\mu\nu\alpha\beta} = i(m_\alpha \bar{m}_\beta - m_\beta \bar{m}_\alpha)\,, \tag{56a}$$

$$m^\mu \bar{m}^\nu \epsilon_{\mu\nu\alpha\beta} = i(k_\alpha l_\beta - k_\beta l_\alpha)\,, \tag{56b}$$

$$k^\mu \bar{m}^\nu \epsilon_{\mu\nu\alpha\beta} = i(k_\alpha \bar{m}_\beta - k_\beta \bar{m}_\alpha)\,, \tag{56c}$$

$$l^\mu m^\nu \epsilon_{\mu\nu\alpha\beta} = i(l_\alpha m_\beta - l_\beta m_\alpha)\,, \tag{56d}$$

and their complex conjugates. The metric tensor of Eq. (22) can be expressed in terms of our null quadruple as

$$g_{\mu\nu} = k_\mu l_\nu + k_\nu l_\mu - m_\mu \bar{m}_\nu - m_\nu \bar{m}_\mu\,. \tag{57}$$

If we now select as the equation for our initial surface Eq. (23), much of the discussion can parallel that of Sec. IV. In particular, we retain the commutation relations of Eqs. (49) and (50), where now the surface element is that of Eq. (24), namely,

$$dS_\rho = k_\rho r^2 \sin\theta dv d\theta d\phi\,. \tag{58}$$

It is no longer true that α_ρ and β_ρ can both be chosen arbitrarily on the initial null surface and continue to satisfy Eq. (45). For, if we examine the components of Eq. (45) when projected on the complete set of bivectors $k^{[\mu}l^{\nu]}$, $l^{[\mu}m^{\nu]}$, $l^{[\mu}\bar{m}^{\nu]}$, $m^{[\mu}\bar{m}^{\nu]}$, $k^{[\mu}m^{\nu]}$, $k^{[\mu}\bar{m}^{\nu]}$, we see from Eq. (56c) and its complex conjugate that two of the resulting equations contain derivatives of α_ρ and β_ρ entirely within the initial surface. These two equations may be written

$$k^\mu \bar{m}^\nu[(\alpha_\mu + i\beta_\mu)_{,\nu} - (\alpha_\nu + i\beta_\nu)_{,\mu}] = 0\,, \tag{59a}$$

$$k^\mu \bar{m}^\nu[(\alpha_\mu - i\beta_\mu)_{,\nu} - (\alpha_\nu - i\beta_\nu)_{,\mu}] = 0\,. \tag{59b}$$

If we regard α_μ and β_μ as real vector fields, Eq. (59b) is evidently the complex conjugate of Eq. (59a). However, it will be convenient for the subsequent development to allow α_μ and β_μ to assume complex values, in which case Eqs. (59a) and (59b) are independent conditions. The remaining four equations, which give the propagation off the initial surface, are

$$(k^\mu l^\nu + m^\mu \bar{m}^\nu)[(\alpha_\mu + i\beta_\mu)_{,\nu} - (\alpha_\nu + i\beta_\nu)_{,\mu}] = 0\,, \tag{60a}$$

$$l^\mu m^\nu[(\alpha_\mu + i\beta_\mu)_{,\nu} - (\alpha_\nu + i\beta_\nu)_{,\mu}] = 0\,, \tag{60b}$$

$$(k^\mu l^\nu - m^\mu \bar{m}^\nu)[(\alpha_\mu - i\beta_\mu)_{,\nu} - (\alpha_\nu - i\beta_\nu)_{,\mu}] = 0\,, \tag{60c}$$

$$l^\mu \bar{m}^\nu[(\alpha_\mu - i\beta_\mu)_{,\nu} - (\alpha_\nu - i\beta_\nu)_{,\mu}] = 0\,. \tag{60d}$$

Equations (60a) and (60b) propagate the components $k^\mu(\alpha_\mu + i\beta_\mu)$ and $m^\mu(\alpha_\mu + i\beta_\mu)$, respectively, off the initial surface. If we differentiate Eq. (59a) in the l^μ direction, we obtain an equation for the propagation of $\bar{m}^\mu(\alpha_\mu + i\beta_\mu)_{,\sigma}k^\sigma$ indicating the usual lack of uniqueness typical of propagation off a characteristic surface. [The remaining undetermined component, $l^\mu(\alpha_\mu + i\beta_\mu)$, reflects our ability to perform an arbitrary infinitesimal gauge transformation and can, without loss of generality, be taken to be everywhere 0.] Similar considerations employing Eqs. (60c), (60d), and (59b) show that the propagation of $k^\mu(\alpha_\mu - i\beta_\mu)$, $\bar{m}^\mu(\alpha_\mu - i\beta_\mu)$, and $m^\mu(\alpha_\mu - i\beta_\mu)_{,\alpha}k^\alpha$ off the initial surface are determined solely by the data on the initial surface. [As before, $l^\mu(\alpha_\mu - i\beta_\mu)$ may also be set equal to zero by a gauge transformation.] Thus we can assert that if we can find a set of initial data which satisfies the two constraint equations (59), a full solution of Eq. (45) can be determined subject to the usual ambiguity typical of propagation off a null surface, as well as the freedom to perform arbitrary gauge transformations. The essential point is that apart from Eqs. (59), there are no further constraints.

Let us consider

$$\alpha_\mu + i\beta_\mu = \alpha \bar{m}_\mu\,, \qquad \alpha_\mu - i\beta_\mu = 0\,, \tag{61}$$

where α is an arbitrary scalar distribution.

Equations (59) are trivially satisfied for any scalar α. The resulting values for the vector fields,

$$\alpha_\mu = \tfrac{1}{2}\alpha \bar{m}_\mu\,, \quad \beta_\mu = -\tfrac{1}{2}i\alpha \bar{m}_\mu\,, \tag{62}$$

when inserted in the integral of Eq. (49) yields, via Eqs. (58) and (56c),

$$\int \frac{\alpha}{2}\bar{m}_\mu k_\nu (F^{\mu\nu} - iF^{\mu\nu *}) r^2 \sin\theta dv d\theta d\phi$$

$$= \int \alpha \bar{m}_\mu k_\nu F^{\mu\nu} r^2 \sin\theta dv d\theta d\phi$$

$$\equiv -\int \alpha \psi r^2 \sin\theta dv d\theta d\phi\,, \tag{63}$$

where ψ is the Penrose function[1] for the Maxwell field, whose form on the null cone fully determines the field in the interior of the cone. In view of the arbitrariness of the scalar field α, a particularly convenient choice will be

$$\alpha = \delta(v - v')\delta(\Omega - \Omega')/rr'\,. \tag{64}$$

We can trivially perform the indicated integrations of Eq. (63) and obtain simply $-\psi(v',\Omega')$. Gathering the terms of Eqs. (62), (63), and (64) in the expression for the commutator, Eq. (49), we thereby obtain the commutator between the field strength $F_{\mu\nu}$, and the Penrose function. Rather than write this out in detail, a cumbersome but straightforward procedure, it will be more illuminating and relevant to determine the commutators for the Penrose function at various points on the null cone. This is readily obtained by multiplying Eq. (49) by $k^\mu \bar{m}^\nu$, and $k^\mu m^\nu$. The form of α_μ given in Eq. (62) yields immediately that $k^\mu \bar{m}^\nu(\alpha_{\mu,\nu} - \alpha_{\nu,\mu}) = 0$. Thus we find

$$[\psi(v,\Omega), \psi(v',\Omega')] = 0\,. \tag{65}$$

From Eq. (65) we obtain by complex conjugation

$$[\bar{\psi}(v,\Omega), \bar{\psi}(v',\Omega')] = 0\,. \tag{66}$$

A less trivial expression is obtained by multiplying

Eq. (49) by $k^\mu m^\nu$. Now we must make explicit use of Eq. (64), as well as Eq. (54). In this manner we find

$$[\bar{\psi}(v,\Omega),\psi(v',\Omega')]=\delta'(v-v')\delta(\Omega-\Omega')/2rr', \quad (67)$$

where $\delta'(v-v')$ is the first derivative of the Dirac delta function. Equations (65), (66), and (67) fully characterize the commutation relations for the Maxwell field on a null cone. In order to obtain the commutation relations on the null cone at infinity we proceed as in Sec. IV. In fact, if we recall the discussion in the paragraph which preceeds Eq. (37), we see that it is convenient to define

$$\rho(v,\Omega)\equiv\lim_{k\to-\infty} r\psi(v,\Omega). \quad (68)$$

In terms of ρ and its complex conjugate $\bar{\rho}$, the commutation relations for the Maxwell field at "past null infinity" are easily seen to be

$$[\rho(v,\Omega),\rho(v',\Omega')]=[\bar{\rho}(v,\Omega),\bar{\rho}(v',\Omega')]=0 \quad (69)$$

and

$$[\rho(v,\Omega),\bar{\rho}(v',\Omega')]=+\tfrac{1}{2}\delta'(v-v')\delta(\Omega-\Omega'). \quad (70)$$

The commutation relations for the Maxwell field at "future null infinity" can be obtained in a fashion strictly parallel to that outlined for the scalar field at the end of Sec. IV and need not be repeated here.

In order to facilitate comparison with the results of Sec. IV, it is convenient to introduce a gauge such that the vector potential A_μ, satisfies an "out-going radiation condition,"

$$k^\mu A_\mu=0. \quad (71)$$

In this gauge, the two components of the vector potential, which characterize the two independent states of polarization of the Maxwell field, may be represented by the single, complex scalar $\bar{m}^\mu A_\mu$. If we define

$$A=r\bar{m}^\mu A_\mu \quad (72)$$

a simple computation, employing Eq. (71), shows that

$$k^\alpha A_{,\alpha}=r\psi. \quad (73)$$

Equations (66) and (67) can therefore be written

$$\left[\frac{\partial A(v,\Omega)}{\partial v},\frac{\partial A(v',\Omega')}{\partial v'}\right]=0, \quad (74)$$

$$\left[\frac{\partial \bar{A}(v,\Omega)}{\partial v},\frac{\partial A(v',\Omega')}{\partial v'}\right]=\tfrac{1}{2}\delta'(v-v')\delta(\Omega-\Omega'). \quad (75)$$

If we therefore require that

$$[A(v,\Omega),A(v',\Omega)]=0 \quad (76)$$

and

$$[\bar{A}(v,\Omega),A(v',\Omega')]=-\tfrac{1}{2}\sigma(v-v')\delta(\Omega-\Omega'), \quad (77)$$

we obtain a set of commutation relations which are equivalent to the original relations (66) and (67). Defining the "news function"

$$a=\lim_{r\to\infty}A=\lim_{r\to\infty}r\bar{m}^\mu A_\mu, \quad (78)$$

we evidently have, analogous to Eq. (73),

$$k^\alpha a_{,\alpha}=\rho. \quad (79)$$

Thus requiring

$$[a(v,\Omega),a(v',\Omega')]=0 \quad (80)$$

and

$$[\bar{a}(v,\Omega),a(v',\Omega')]=-\tfrac{1}{2}\sigma(v-v')\delta(\Omega-\Omega'), \quad (81)$$

we obtain a set of commutation relations which are equivalent to the original relations (69) and (70).

Equations (80) and (81) are exactly of the form that we should expect from the results of Sec. IV, and are presented for purposes of comparison with that section. However, it is evident that they are not the only possibility, at least in so far as the derivation presented in this paper is concerned. For in order to obtain them, two integrations had to be performed, one with respect to v and one with respect to v'. Even with reasonable boundary conditions, which would exclude terms which diverge for large v and v', we are still left with the addition of an arbitrary antisymmetric function of the angles to each of the commutators (80), (81), or (76), (77). These surely ought to vanish in view of the space-like character of the pair of points concerned. Nor is the situation substantially improved by Eq. (48) in the radiation gauge, for an integration with respect to v' would still be required in order to obtain the commutation relations among the transverse components of the potentials. Perhaps an argument similar to that which led to Eq. (33) can be found which could exclude these arbitrary functions.

VI. LINEARIZED GRAVITATION THEORY

The linearized theory of gravitation may be obtained from the Einstein theory of general relativity by a process of discarding all nonlinear terms, or it may be presented as an independent field theory in its own right. We prefer to take the latter course in order to parallel the presentations of the preceding sections. [Recall that throughout this section we shall maintain the notation that a comma shall denote covariant differentiation with respect to the flat background metric which in polar coordinates we will take to have the form of Eq. (22).]

The potential of the gravitational field is represented by a symmetric tensor, $h_{\mu\nu}$, which is not directly an observable, but is subject to gauge transformations such that tensors which can be obtained from one another by the relation

$$h_{\mu\nu}=h_{\mu\nu}-\gamma_{\mu,\nu}-\gamma_{\nu,\mu}, \quad (82)$$

where γ_μ is an arbitrary vector field, are understood to describe the same gravitational field. The gauge-invariant field quantity, analogous to $F_{\mu\nu}$ in the case of the Maxwell field is the fourth order tensor

$$R_{\alpha\beta\gamma\delta}\equiv\tfrac{1}{2}(h_{\alpha\delta,\beta\gamma}+h_{\beta\gamma,\alpha\delta}-h_{\beta\delta,\alpha\gamma}-h_{\alpha\gamma,\beta\delta}), \qquad (83)$$

which has all the symmetries well known for the Riemann tensor of Riemannian geometry. It will be convenient for purposes of ultimate comparison with the full Einstein theory to define the following tensors

$$\Gamma_{\beta\gamma}{}^{\alpha}\equiv\tfrac{1}{2}g^{\alpha\mu}(h_{\mu\beta,\gamma}+h_{\mu\gamma,\beta}-h_{\beta\gamma,\mu}), \qquad (84)$$

$$R_{\alpha\beta}\equiv g^{\mu\nu}R_{\alpha\mu\beta\nu}, \quad R\equiv g^{\alpha\beta}R_{\alpha\beta}, \qquad (85)$$

$$G_{\alpha\beta}\equiv R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R, \qquad (86)$$

$$C_{\alpha\beta\gamma\delta}\equiv R_{\alpha\beta\gamma\delta}-\tfrac{1}{2}(g_{\alpha\gamma}R_{\beta\delta}+g_{\beta\delta}R_{\alpha\gamma}-g_{\beta\gamma}R_{\alpha\delta}-g_{\alpha\delta}R_{\beta\gamma}) -\tfrac{1}{6}(g_{\beta\gamma}g_{\alpha\delta}-g_{\alpha\gamma}g_{\beta\delta})R. \qquad (87)$$

As the notation implies, the above quantities may be obtained by the process of linearization of the corresponding quantities of the full theory. Equation (83) corresponds to the Riemann or curvature tensor, Eq. (84) to the Christoffel symbols or affine connection, Eq. (85) to the Ricci tensor and Ricci scalar, Eq. (86) to the Einstein tensor, and Eq. (87) to the Weyl or conformal curvature tensor. In addition to having all the symmetries of the Riemann tensor, $C_{\alpha\beta\gamma\delta}$ has vanishing trace on each pair of indices. We note that as a consequence of Eq. (83) the Einstein tensor can be shown to satisfy the identity

$$G^{\mu}{}_{\alpha,\mu}=0. \qquad (88)$$

The Lagrangian for the linearized gravitation theory is

$$L=g^{\mu\nu}(\Gamma^{\alpha}{}_{\mu\nu}\Gamma^{\beta}{}_{\alpha\beta}-\Gamma^{\beta}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\beta}), \qquad (89)$$

from which we can deduce the field equations

$$G_{\alpha\beta}=0. \qquad (90)$$

Let us now introduce an auxiliary tensor field, $\alpha_{\mu\nu\sigma}$, which is defined to have the following symmetries

$$\begin{gathered}\alpha_{\mu\nu\sigma}=-\alpha_{\nu\mu\sigma}, \quad \alpha^{\mu}{}_{\nu\mu}=0\\ \alpha_{\mu\nu\sigma}+\alpha_{\nu\sigma\mu}+\alpha_{\sigma\mu\nu}=0\\ \left\{\begin{array}{ll}\alpha_{\mu\nu\sigma}n^{\sigma}=0 & \text{for the space-like case}\\ \alpha_{\mu\nu\sigma}l^{\sigma}=0 & \text{for the null case}\end{array}\right\}.\end{gathered} \qquad (91)$$

(n^{σ} being the unit normal to the family of space-like hypersurfaces employed.)

It follows from these relations that $\alpha_{\mu\nu\sigma}$ has precisely ten independent components. With the aid of this tensor we can now define the vector field

$$C^{\rho}\equiv\alpha_{\mu\nu\sigma}C^{\mu\nu\sigma\rho}+\alpha^{\rho}{}_{\mu\nu}G^{\mu\nu}+2\gamma_{\mu}G^{\mu\rho}, \qquad (92)$$

where γ_{μ} is an arbitrary vector field. Taking the divergence of C^{ρ}, we find after some rearrangement of terms and employing the identity Eq. (88),

$$C^{\rho}{}_{,\rho}+(-\alpha^{\mu}{}_{\sigma\rho,\mu}-2\gamma_{\sigma,\rho})G^{\sigma\rho}=\alpha_{\mu\nu\sigma,\rho}C^{\mu\nu\sigma\rho}. \qquad (93)$$

In analogy with Eqs. (15) and (45), if we require the auxiliary field to satisfy the differential equations

$$\alpha_{\mu\nu\sigma,\rho}-\alpha_{\mu\nu\rho,\sigma}+\alpha_{\sigma\rho\mu,\nu}-\alpha_{\sigma\rho\nu,\mu}-\tfrac{1}{2}[g_{\mu\rho}(\alpha^{\gamma}{}_{\nu\sigma,\gamma}+\alpha^{\gamma}{}_{\sigma\nu,\gamma}) +g_{\nu\sigma}(\alpha^{\gamma}{}_{\rho\mu,\gamma}+\alpha^{\gamma}{}_{\mu\rho,\gamma})-g_{\nu\rho}(\alpha^{\gamma}{}_{\mu\sigma,\gamma}+\alpha^{\gamma}{}_{\sigma\mu,\gamma}) -g_{\mu\sigma}(\alpha^{\gamma}{}_{\rho\nu,\gamma}+\alpha^{\gamma}{}_{\nu\rho,\gamma})]=0 \qquad (94)$$

the right-hand side of Eq. (93) will vanish identically.

In this fashion we can satisfy Eq. (12) with

$$\delta h_{\mu\nu}=-\tfrac{1}{2}(\alpha^{\gamma}{}_{\mu\nu,\gamma}+\alpha^{\gamma}{}_{\nu\mu,\gamma})-\gamma_{\mu,\nu}-\gamma_{\nu,\mu}. \qquad (95)$$

[We note at this point that Eq. (94) assures that $\delta h_{\mu\nu}$ satisfies the field equations (90).]

A comparison with Eq. (82) indicates that the term in C^{ρ} containing γ_{μ}, namely, $2\gamma_{\mu}G^{\mu\rho}$, generates a pure gauge transformation. As in the previous section, we can either exploit the freedom to perform gauge transformations in order to establish a preferred gauge frame, or we can work exclusively with gauge-invariant quantities. We shall again prefer to take the latter course.

We now conclude from Eqs. (4), (92), and (95)

$$\left[h_{\mu\nu}, \int\alpha_{\alpha\beta\gamma}C^{\alpha\beta\gamma\rho}dS_{\rho}\right]=-\tfrac{1}{2}(\alpha^{\gamma}{}_{\mu\nu,\gamma}+\alpha^{\gamma}{}_{\nu\mu,\gamma}) -\gamma_{\mu,\nu}-\gamma_{\nu,\mu}, \qquad (96)$$

where we have taken note of the field equations, Eq. (90), to discard the last two terms in the definition of C^{ρ}, Eq. (92). Differentiating Eq. (96) we obtain the gauge-invariant commutation relations

$$\left[C_{\alpha\beta\gamma\delta}, \int\alpha_{\mu\nu\sigma}C^{\mu\nu\sigma\rho}dS_{\rho}\right]=-\tfrac{1}{4}(\alpha^{\mu}{}_{\alpha\delta,\mu\beta\gamma}+\alpha^{\mu}{}_{\beta\gamma,\mu\alpha\delta} +\alpha^{\mu}{}_{\delta\alpha,\mu\beta\gamma}+\alpha^{\mu}{}_{\gamma\beta,\mu\alpha\delta}-\alpha^{\mu}{}_{\alpha\gamma,\mu\beta\delta}-\alpha^{\mu}{}_{\beta\delta,\mu\alpha\gamma} -\alpha^{\mu}{}_{\gamma\alpha,\mu\beta\delta}-\alpha^{\mu}{}_{\delta\beta,\mu\alpha\gamma}). \qquad (97)$$

[We have ignored the distinction between $R_{\alpha\beta\gamma\delta}$ and $C_{\alpha\beta\gamma\delta}$ in view of Eq. (90).]

The auxiliary field, $\alpha_{\mu\nu\sigma}$, of course must satisfy the differential equations (94). If we confine our attention to obtaining commutation relations on an initial space-like hypersurface, the situation is relatively simple. A careful inspection indicates that Eq. (94) consists of ten independent linear first-order equations which uniquely determine the first derivative normal to the hypersurface of the ten independent components of $\alpha_{\mu\nu\sigma}$, as linear combinations of first derivatives within the space-like hypersurface. It follows that $\alpha_{\mu\nu\sigma}$ may be assigned arbitrarily on the initial hypersurface and Eq. (94), together with the symmetry conditions, Eqs. (91), will determine it uniquely everywhere. In order to evaluate the right-hand side of Eq. (97), explicit use will have to be made of Eq. (94). Although the procedure is entirely straightforward, the resulting expressions become rather complicated and are not very illuminating. We therefore leave to the amusement of the interested reader the determination of the specific expressions for the commutation relations of the various components of the curvature tensor on an initial spacelike surface.

For the case where the surface of integration in Eq. (97) is taken to be null, the analysis of the initial value problem for Eq. (94) proceeds somewhat differently. Let us for simplicity call the left-hand side of Eq. (94) $\sigma_{\mu\nu\sigma\rho}$. (We note that $\sigma_{\mu\nu\sigma\rho}$ has all the symmetries of the Weyl tensor.) If we resolve the components of $\sigma_{\mu\nu\sigma\rho}$ relative to the null quadruple, Eq. (55), the ten independent equations, (94), separate into two groups, in the following fashion

$$\begin{aligned} \bar{m}^\alpha k^\beta \bar{m}^\gamma k^\delta \sigma_{\alpha\beta\gamma\delta} &= 0\,, \\ m^\alpha k^\beta m^\gamma k^\delta \sigma_{\alpha\beta\gamma\delta} &= 0\,, \end{aligned} \tag{98}$$

and

$$\begin{aligned} \bar{m}^\alpha k^\beta l^\gamma k^\delta \sigma_{\alpha\beta\gamma\delta} &= 0\,, \\ m^\alpha k^\gamma l^\gamma k^\delta \sigma_{\alpha\beta\gamma\delta} &= 0\,, \\ l^\alpha k^\beta l^\gamma k^\delta \sigma_{\alpha\beta\gamma\delta} &= 0\,, \\ \bar{m}^\alpha m^\beta l^\gamma k^\delta \sigma_{\alpha\beta\gamma\delta} &= 0\,, \\ m^\alpha l^\beta l^\gamma k^\delta \sigma_{\alpha\beta\gamma\delta} &= 0\,, \\ \bar{m}^\alpha l^\beta l^\gamma k^\delta \sigma_{\alpha\beta\gamma\delta} &= 0\,, \\ m^\alpha l^\beta m^\gamma k^\delta \sigma_{\alpha\beta\gamma\delta} &= 0\,, \\ \bar{m}^\alpha l^\beta \bar{m}^\gamma l^\delta \sigma_{\alpha\beta\gamma\delta} &= 0\,. \end{aligned} \tag{99}$$

Although many equations appear to occur as complex conjugate pairs, this is not, in fact, the case since we shall find it convenient to take $\sigma_{\mu\nu\sigma\rho}$ to be a complex tensor field. The reason for distinguishing between Eqs. (98) and (99), is that Eq. (98) is found to contain only derivatives of $\alpha_{\mu\nu\sigma}$ entirely within the null surface, Eq. (23), whereas Eq. (99) all contain derivatives off the null surface. One can check that Eqs. (99) propagate off the initial null surface all the independent components of $\alpha_{\mu\nu\sigma}$ with the exception of $k^\mu \bar{m}^\nu \bar{m}^\sigma \alpha_{\mu\nu\sigma}$ and $k^\mu m^\nu m^\sigma \alpha_{\mu\nu\sigma}$.

For the determination of the propagation of these latter two components we must addend to these set of equations the two obtained by differentiating Eqs. (98) in the l^μ direction:

$$\begin{aligned} \bar{m}^\alpha k^\beta \bar{m}^\gamma k^\delta l^\rho \sigma_{\alpha\beta\gamma\delta,\rho} &= 0\,, \\ m^\alpha k^\beta m^\gamma k^\delta l^\rho \sigma_{\alpha\beta\gamma\delta,\rho} &= 0\,. \end{aligned} \tag{100}$$

Equations (100) will now determine the propagation of $k^\mu \bar{m}^\nu \bar{m}^\sigma \alpha_{\mu\nu\sigma,\rho} k^\rho$ and $k^\mu m^\nu m^\sigma \alpha_{\mu\nu\sigma,\rho} k^\rho$ off the initial null surface. We can therefore assert that if we can find a set of initial data which satisfies the two constraint equations, (98), a full solution of Eq. (94) can be determined subject to the usual ambiguity typical of propagation off a null surface.

Let us now consider

$$\alpha_{\mu\nu\sigma} \equiv \tfrac{1}{2}\alpha(\bar{m}_\mu k_\nu - \bar{m}_\nu k_\mu)\bar{m}_\sigma\,, \tag{101}$$

where α is an arbitrary scalar distribution. It is evident that Eq. (101) satisfies all the required symmetries, Eq. (91). A little computation confirms that Eq. (101) also identically satisfies the constraints, Eqs. (98), for arbitrary choice of α on the initial null surface. It is of course not permissible to assume that $\alpha_{\mu\nu\sigma}$ continues to have the form Eq. (101) off the initial surface, for the propagation off the surface is governed by the remaining equations, (99) and (100). [In view of the terms of the form $\alpha^\mu{}_{\sigma\rho,\mu}$, which appear on the right-hand side of Eq. (96), we shall in fact have to take these remaining equations into consideration in our subsequent calculations, in contrast to the procedure employed at the corresponding point of the previous section.]

If we again take the surface element of the integral in Eq. (97) to be given by Eq. (58), and the arbitrary scalar distribution, α, to be given by Eq. (64), the left-hand side of Eq. (97) assumes the form $[C_{\alpha\beta\gamma\delta}(v,\Omega), \psi(v',\Omega)]$, where we have introduced the Penrose function for the gravitational field

$$\psi \equiv \bar{m}^\alpha k^\beta \bar{m}^\gamma k^\delta C_{\alpha\beta\gamma\delta}\,. \tag{102}$$

If we are primarily concerned with the determination of commutation relations for the Penrose function and its complex conjugate, $\bar{\psi}$, on the initial null surface, it will not be necessary to use all of the components of Eq. (97). It is sufficient to consider the two expressions obtained by multiplying Eq. (97) by $\bar{m}^\alpha k^\beta \bar{m}^\gamma k^\delta$ and $m^\alpha k^\beta m^\gamma k^\delta$. Upon performing these operations and substituting Eqs. (101) and (64) into the right-hand side of Eq. (97), we obtain after some tedious but rather straightforward computations

$$[\psi(v,\Omega),\psi(v',\Omega')] = 0 \tag{103}$$

and

$$[\bar{\psi}(v,\Omega),\psi(v',\Omega')] = -\frac{1}{4}\frac{\delta'''(v-v')}{rr'}\delta(\Omega-\Omega')\,, \tag{104}$$

where $\delta'''(v-v')$ is the third derivative of the Dirac delta function. Equations (103) and (104) fully characterize the commutation relations of the linearized gravitational field on null cone given by Eq. (23). As in the previous theories, if we desire to obtain the commutation relations on the null surface at the "past infinity" it is again convenient to define $\rho(v,\Omega)$ as in Eq. (68).

It then follows immediately from Eqs. (103) and (104)

$$[\rho(v,\Omega),\rho(v',\Omega')] = 0 \tag{105}$$

and

$$[\bar{\rho}(v,\Omega),\rho(v',\Omega')] = -\tfrac{1}{4}\delta'''(v-v')\delta(\Omega-\Omega')\,. \tag{106}$$

The commutation relations for the linearized gravitational field at "future null infinity" can be obtained in an analogous fashion as indicated at the end of Sec. IV.

In order to facilitate a comparison with the results of Secs. IV and V it is convenient to introduce a gauge such that the potential $h_{\mu\nu}$ satisfies an "outgoing radiation condition"

$$k^{\mu}h_{\mu\alpha}=0. \tag{107}$$

In this gauge the components of $h_{\mu\nu}$, which characterize the two independent states of polarization of gravitational radiation, can be represented by the single complex scalar $\bar{m}^{\mu}\bar{m}^{\nu}h_{\mu\nu}$. If we define

$$H\equiv(1/\sqrt{2})r\bar{m}^{\mu}\bar{m}^{\nu}h_{\mu\nu}, \tag{108}$$

a simple calculation employing Eq. (107) yields

$$k^{\mu}k^{\nu}H_{,\mu\nu}=\sqrt{2}r\psi. \tag{109}$$

Equations (103) and (104) can therefore be written

$$\left[\frac{\partial^2 H(v,\Omega)}{\partial v^2},\frac{\partial^2 H(v',\Omega')}{\partial v'^2}\right]=0 \tag{110}$$

$$\left[\frac{\partial^2 \bar{H}(v,\Omega)}{\partial v^2},\frac{\partial^2 H(v',\Omega')}{\partial v'^2}\right]=-\tfrac{1}{2}\delta'''(v-v')\delta(\Omega-\Omega'), \tag{111}$$

respectively.

Integrating these last two equations we obtain, modulo the degree of arbitrariness discussed at the conclusion of the previous section

$$[H(v,\Omega),H(v',\Omega')]=0 \tag{112}$$

and

$$[\bar{H}(v,\Omega),H(v',\Omega')]=-\tfrac{1}{2}\sigma(v-v')\delta(\Omega-\Omega'). \tag{113}$$

If we define the gravitational "news function"

$$h\equiv\lim_{r\to\infty}H=\frac{1}{\sqrt{2}}\lim_{r\to\infty}r\bar{m}^{\mu}\bar{m}^{\nu}h_{\mu\nu}, \tag{114}$$

at "past null infinity" we have analogous to Eq. (109)

$$k^{\mu}k^{\nu}h_{,\mu\nu}=\sqrt{2}\rho. \tag{115}$$

Thus, apart from an additive arbitrary antisymmetric function of the angles we can conclude that the relations

$$[h(v,\Omega),h(v',\Omega')]=0 \tag{116}$$

and

$$[\bar{h}(v,\Omega),h(v',\Omega')]=-\tfrac{1}{2}\sigma(v-v')\delta(\Omega-\Omega') \tag{117}$$

are equivalent to Eqs. (105) and (106). We note the striking similarity of these commutation relations and those of the Klein-Gordon and Maxwell fields.

VII. THE GENERAL THEORY OF RELATIVITY

The primary interest in the preceding development lies in the possibility of extending it to the general theory of relativity. A presentation for general relativity, to a large extent parallel to that of Sec. VI, has been in print for some time.[6] Let me briefly review its development in order to indicate the cause of its foundering and to show how the considerations of this paper can repair some of the damage.

The gravitational field is described by a symmetric tensor, $g_{\mu\nu}$, which is not directly observable. Tensors obtainable from one another by means of general curvilinear coordinate transformations are understood to describe the same gravitational field. Nor is the Riemann tensor directly observable in this theory, since it too changes its form under curvilinear coordinate transformations. Only constants of the motion are invariant under general coordinate transformations and represent observables. We shall introduce the usual notation that a semicolon subscript shall denote covariant differentiation with respect to the Christoffel symbols $\Gamma^{\sigma}{}_{\mu\nu}$ determined by the metric $g_{\mu\nu}$. (The fact that there is no background metric with respect to which one can specify once and for the metric and/or affine properties of the manifold is the principle source of the sundry difficulties that we encounter in this theory.)

The field equations satisfied by the metric are given by Eq. (90), where the symbols on the left-hand side of Eqs. (83) through (87) are now to be understood in their usual meaning in Riemannian geometry. If we again define a tensor field $\alpha_{\mu\nu\sigma}$ having the symmetries of Eq. (91) we can employ it as before to construct the vector field, C^{ρ}, as in Eq. (92). Analogous to Eq. (93) we now have

$$C^{\rho}{}_{;\rho}+(-\alpha^{\mu}{}_{\sigma\rho;\mu}-2\gamma_{\sigma;\rho})G^{\sigma\rho}=\alpha_{\mu\nu\sigma;\rho}C^{\mu\nu\sigma\rho}, \tag{118}$$

the principal difference being the occurrence of semicolons now instead of commas. It would be tempting at this point, in analogy with Eq. (94), to require

$$\alpha_{(\mu\nu\sigma;\rho)}=0 \tag{119}$$

(where the parenthesis denotes that the indices are to be symmetrized according to all the symmetries of the Weyl tensor), and thereby conclude the analog of Eq. (95)

$$\delta g_{\mu\nu}=-\tfrac{1}{2}(\alpha^{\sigma}{}_{\mu\nu;\sigma}+\alpha^{\sigma}{}_{\nu\mu;\sigma})-\gamma_{\mu;\nu}-\gamma_{\nu;\mu}. \tag{120}$$

This is precisely how one proceeded in Ref. 6. The principal error of this approach is that in view of the fact that a semicolon appears in Eq. (119) rather than a comma (as well as other metric-dependent terms required to eliminate the trace of the expression), Eq. (119) does not define a class of functions independent of the dynamical field. Thus, although it is still true that, when Eq. (119) is satisfied, $\int C^{\rho}dS_{\rho}$ is a constant of the motion, it is no longer true that what it generates is correctly given by Eq. (120). In effect there are terms on the right-hand side of Eq. (118) which are functionally dependent on the field equations, $G_{\mu\nu}$. This observation is further confirmed by the fact that $\delta g_{\mu\nu}$, given by Eq. (120) does not satisfy the perturbed Einstein field equations as a consequence of Eq. (119). Curvature-dependent terms appear which did not occur in the linearized theory.

[6] P. G. Bergmann and A. B. Komar, Les Theories Relativistes de la Gravitation, Colloques Internationaux du C.N.R.S. XCI, Royaumont, 1959, Éditions du C.N.R.S., Paris, 1962, p. 309.

If, however, we wish to consider a truncated theory, obtained from general relativity by imposing the boundary conditions that the only spaces to be admitted are those which are asymptotically flat in the sense of Penrose,[2] we can recover the validity of some of the relations obtained in the previous section. In fact, if we note that the correct expressions, Eqs. (118) and (119), deviate from the corresponding expressions of linearized theory by terms which vanish in the limit $r \rightarrow \infty$, in this limit, by paralleling the steps of the preceding section, we can rigorously recover the commutation relations Eqs. (105) and (106) or equivalently Eqs. (116) and (117) for the full Einstein theory. (With this hypothesis of Sachs[5] thus established, the reader is referred to his paper for a presentation of how this set of commutation relations may be employed to represent the motion group available to manifold at infinity.)

In view of the fact that the commutation relations at infinity for the news function of the asymptotically flat full theory is identical to those of the linearized theory, the reader may wonder in which way the quantum theories obtained by use of these Poisson brackets could possibly differ. Although the pure incoming fields defined at "past null infinity," and the pure outgoing fields defined at "future null infinity" satisfy identical commutation relations in both theories, the relationship between the incoming and the outgoing fields is vastly different for the two theories, requiring, as it does, an integration from "past null infinity" to "future null infinity" of the field equations of the theory.

In conclusion we would like to make three comments:

(1) It should be possible to derive the commutation relations for the Einstein theory at null infinity by working directly on that surface at infinity, without reference to a limiting procedure so necessary in our present development. This will be done in a subsequent paper by employing the technique of conformal transformations to bring the surface at infinity into a finite region where it can be more easily studied.

(2) One would not expect a quantum theory based on the commutation relations developed for this truncated theory to be equivalent to a quantization of the full nontruncated theory, or for a theory truncated in a different fashion, for example by admitting only solutions of the field equations which are spacially closed and/or simply connected. It does not appear to be possible to extend the methods of this paper to treat these more general spaces. A quantum theory which would admit states of a closed universe would probably have to be constructed by rather different considerations.

(3) The arbitrary antisymmetric functions of the angles which occurred in the expressions for the commutators of the news function are present due to the existence of the possibility of performing Bondi-Metzner transformations[5] at null infinity. We have succeeded in eliminating these functions from the scalar field theory (and we should expect to eliminate them from the Maxwell and linearized gravitation theory) by making explicit reference to the family of null cones used, including the manner of anchoring their vertices. For the full Einstein theory, this does not appear to be possible, and we should therefore not expect to be able to eliminate these arbitrary functions of angle from the expressions for commutators which are equivalent to Eqs. (116) and (117) of the linearized theory. One may therefore prefer to eliminate all reference to such arbitrary functions by working with commutators obtained through differentiating these latter expressions once with respect to either v or v'. That such a procedure can provide commutation relations which are covariant under Bondi-Metzner transformations appears to be intimately related to the proof given by Sachs[5] of the existence of an integrable affine connection at null infinity.

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